

# Magnetism and infrared divergence in a Hubbard-phonon interacting system

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## Abstract

We show that a finite Hubbard-phonon interacting system has a ferromagnetic or unique spin-singlet ground state under the *infrared singular condition*. The key tool is a unitary transformation introduced by Arai and Hirokawa [4]. We construct a concrete infrared singular representation using the operator algebraic method. The method is essentially same as one for the van Hove model using the Wightman functional method [2].

## 1 Introduction

The electron-phonon interacting system is an important model for quantum statistical mechanics and quantum field theory, and is fundamental for condensed matter physics. Nevertheless there are few mathematically rigorous results. For example Freericks and Lieb study this model [16], but they neglect high-energy phonons in absolute zero temperature. They consider phonons quantum mechanically, not quantum field theoretically. Our purpose is to consider phonons field theoretically. We prove mathematically rigorously the physical folklore that interaction between electrons may become attractive if there exists exchange of bosons. We construct models exhibit special magnetic properties. Furthermore they occur even if the model is under the infrared singular condition for phonons.

We have two main results: the first is a statistical mechanical feature, *phase transition* such as ferromagnetism and (some type of) superconductivity. The electron-phonon interacting system exhibits many interesting phenomena: ferromagnetism and superconductivity are typical examples. Freericks and Lieb [16] proves that the ground state is unique and spin-singlet if the interaction between electrons is non-positive. This is an extension of Lieb's famous work [20]. However they do not show the occurrence of the attractive force between electrons when electron-phonon interaction exists. We can prove it in this paper. Of course this has been proved rigorously even if electrons obey the Schrödinger equation [17]. Our result is novel in terms of a 'phase transition between ferromagnetism and superconductivity'. We can construct models which exhibits ferromagnetism if a coupling constant is small and superconductivity if it is large.

The second is a quantum field theoretic feature. Arai and Hirokawa study a general field theoretic model and call it a *GSB (generalized spin-boson) model* [4, 5]. It includes many interesting examples: a usual spin-boson model, point particles with bosonic fields, our Hubbard-phonon model, and so on. Arai, Hirokawa, Hiroshima, and others investigate the general theory of it [1, 4, 5, 7, 8, 9]. Their studies reveal the general structure, and hence we would like to investigate concrete models according to the spirit of constructive quantum field theory.

In cited studies the main concern is (non-)existence of the ground state. Let  $\omega$  be a dispersion relation for bosons and  $\sigma(\omega)$  be its spectrum. Bosons are called *massless* if  $\inf \sigma(\omega) = 0$  and are *massive* otherwise. If bosons are massless we face a notorious *infrared divergence* problem. Hence it is a natural task to investigate (non-)existence. We prove the existence of ground states under the infrared singular condition.

On the other hand, as far as the author knows, the uniqueness problem has not been systematically studied. In [4, 5] Arai and Hirokawa study several examples and ground states are unique or non-unique in those examples: it may be highly model dependent and the physical reason for non-uniqueness is not clear. In non-relativistic QED, a related model, Hiroshima and Spohn prove the existence of two-fold degenerated ground states if an electron is with spin [18]. However this non-uniqueness is physically trivial, i.e., spin orientation. Hence we would like to construct examples having many ground states with physically clear meaning. If the model exhibits a phase transition we may have many degenerated ground states. Thus our target is a phase transition, especially ferromagnetism.

Outline of this paper is as follows. First we fix notations and introduce an important unitary operator in [4]. This unitary makes our analysis very easy. Next we construct models which exhibit special magnetic properties under the infrared regularization. We can use various interesting results for analysis of finite Hubbard models [20, 28]. This is also crucial. Finally we remove infrared cutoff. The technique is essentially same as for the van Hove model [2]. Under the infrared singular condition the model also exhibit various magnetic properties. The study of an infinite Hubbard-phonon system is in progress. We have many fundamental and interesting problems.

## 2 Mathematical settings and a unitary transformation

Suppose  $\Lambda$  is a finite set and  $\mathcal{K}$  is a separable Hilbert space. Our Hilbert spaces for the model are as follows:

$$\mathcal{F} := \mathcal{F}_e \otimes \mathcal{F}_b, \quad (2.1)$$

$$\mathcal{F}_e := \bigoplus_{n=1}^{|\Lambda|} \bigotimes_{\text{as}}^n \ell^2(\Lambda; \mathbb{C}^2), \quad (2.2)$$

$$\mathcal{F}_b := \bigoplus_{n=1}^{\infty} \bigotimes_s^n \mathcal{K}, \quad (2.3)$$

where  $\oplus$  is a direct sum,  $\otimes$  is a tensor product, and subscripts s/as mean symmetric/anti-symmetric tensor. For a subspace  $\mathcal{D} \subset \mathcal{K}$  we define a subspace

$$\mathcal{F}_{b,\text{fin}}(\mathcal{D}) := \bigoplus_{n=0}^{\infty} \bigotimes_s^n \mathcal{D}, \quad (2.4)$$

where  $\hat{\oplus}_{n=0}^{\infty}$  (resp.  $\hat{\otimes}_s^n$ ) is an algebraic direct sum (resp. algebraic tensor product).

Our Hamiltonians are

$$H := H_{\text{fr}} + \alpha H_{\text{I}}, \quad (2.5)$$

$$H_{\text{fr}} := H_e \otimes 1 + 1 \otimes H_b, \quad (2.6)$$

$$H_e := d\Gamma_e(T) + U \sum_{x \in \Lambda} n_{x,+} n_{x,-}, \quad n_{x,s} := c_{x,s}^* c_{x,s}, \quad (2.7)$$

$$H_b := d\Gamma_b(\omega), \quad (2.8)$$

$$H_{\text{I}} := \sum_{x \in \Lambda} n_x \otimes \phi(\lambda_x), \quad n_x = n_{x,+} + n_{x,-}, \quad (2.9)$$

where  $\alpha \in \mathbb{R}$  is a coupling constant,  $T = (t_{x,y})_{x,y \in \Lambda}$  is a self-adjoint operator on  $\ell^2(\Lambda; \mathbb{C}^2)$ ,  $U > 0$  is a Coulomb repulsion,  $d\Gamma_{b/e}$  is a boson/fermion second quantization,  $\omega$  is a non-negative self-adjoint operator on  $\mathcal{K}$  (a dispersion relation for phonons). The operator  $c_{x,s}^{\#}$  is a creation/annihilation operator for electrons obeying the CAR (canonical anti-commutation relation):

$$\{c_{x,s}, c_{y,t}\} = \{c_{x,s}^*, c_{y,t}^*\} = 0, \quad \{c_{x,s}, c_{y,t}^*\} = \delta_{x,y} \delta_{s,t}, \quad (2.10)$$

where  $\{A, B\} := AB + BA$ . Finally  $\phi(\lambda)$  is Segal's field operator and vectors  $\{\lambda_x\}$  are in  $\mathcal{K}$ .

For simplicity we assume  $\omega$  has no eigenvalues. Denote  $E_0(A) := \inf \sigma(A)$  for a self-adjoint operator  $A$ , and call it a ground state energy of  $A$ .

Here we impose some properties on cutoff vectors  $\{\lambda_x\}$ .

**Assumption. 2.1** *Vectors  $\{\lambda_x\}$  are in  $\text{dom } \omega^{-1/2}$ . Any partial sum of  $\{\lambda_x\}$  is not in  $\text{dom } \omega^{-1}$ . Furthermore they are "real" vectors in the sense that*

$$\langle \lambda_x, \lambda_y \rangle, \quad \langle \omega^{-1/2} \lambda_x, \omega^{-1/2} \lambda_y \rangle \in \mathbb{R}, \quad \forall x, y \in \Lambda. \blacksquare \quad (2.11)$$

The condition  $\lambda_x \notin \text{dom } \omega^{-1}$  is said to be the *infrared singular condition*. In the following we always impose the above condition.

In the Fock representation of CCR (canonical commutation relation), we have the following useful inequality:

$$\|\phi(\lambda_x) \Psi_b\| \leq \frac{1}{\sqrt{2}} \left( 2 \|\omega^{-1/2} \lambda_x\| \|H_b^{1/2} \Psi_b\| + \|\lambda_x\| \|\Psi_b\| \right), \quad \Psi_b \in \text{dom } H_b^{1/2}. \quad (2.12)$$

Then we have

**Theorem. 2.1** [4]  *$H$  is essentially self-adjoint on  $\mathcal{D}_H := \mathcal{F}_e \hat{\otimes} \mathcal{F}_{b,\text{fin}}(\text{dom } \omega)$ , and bounded from below.*

(Proof) Let  $\Psi \in \mathcal{D}_H$ . Then

$$\|\alpha H_{\text{I}} \Psi\| \leq |\alpha| \sum_{x \in \Lambda} \|(n_x \otimes 1)(1 \otimes \phi(\lambda_x)) \Psi\| \quad (2.13)$$

$$\leq |\alpha| \sum_{x \in \Lambda} \frac{1}{\sqrt{2}} \left( 2 \|\omega^{-1/2} \lambda_x\| \|1 \otimes H_b^{1/2} \Psi\| + \|\lambda_x\| \|\Psi\| \right). \quad (2.14)$$

Put  $\tilde{H}_{\text{fr}} := (H_e - E_0(H_e)) \otimes 1 + 1 \otimes H_b \geq 0$ . For any  $\varepsilon > 0$ , using

$$\|1 \otimes H_b^{1/2} \Psi\| \leq \|1 \otimes H_b \Psi\|^{1/2} \|\Psi\|^{1/2} \leq \varepsilon \|\tilde{H}_{\text{fr}} \Psi\| + \frac{1}{4\varepsilon} \|\Psi\|, \quad (2.15)$$

we obtain

$$\|\alpha H_I \Psi\| \leq \varepsilon \left( \sqrt{2} |\alpha| \sum_{x \in \Lambda} \|\omega^{-1/2} \lambda_x\| \right) \|\tilde{H}_{\text{fr}} \Psi\| + |\alpha| \sum_{x \in \Lambda} \left( \frac{1}{\sqrt{2}} \|\lambda_x\| + \frac{\sqrt{2}}{4\varepsilon} \|\omega^{-1/2} \lambda_x\| \right) \|\Psi\|. \quad (2.16)$$

Since  $\mathcal{D}_H$  is a core for  $\text{dom } \tilde{H}_{\text{fr}}$ , the above inequality holds for any  $\Psi \in \text{dom } \tilde{H}_{\text{fr}}$ . Moreover since  $\varepsilon > 0$  is arbitrary, there exists an  $\varepsilon > 0$  satisfying  $\varepsilon (\sqrt{2} |\alpha| \sum_{x \in \Lambda} \|\omega^{-1/2} \lambda_x\|) < 1$ . Hence Kato-Relich's theorem establishes the result. ■

**Remark. 2.2** *If we impose some summability conditions on  $\{\lambda_x\}$ , we can prove self-adjointness of  $H$  for infinite  $\Lambda$ . However this condition may be unphysical. See the Remark 4.2.*

For the time being we impose infrared cutoff.

**Assumption. 2.2** *Assume  $\lambda_x^\kappa \in \text{dom } \omega^{-1/2} \cap \text{dom } \omega^{-1}$  for  $\kappa > 0$  (this is called infrared regularization). They also satisfy the “real” vector condition. See the Assumption 2.1. Moreover  $\lambda_x^\kappa \rightarrow \lambda_x$  and  $\omega^{-1/2} \lambda_x^\kappa \rightarrow \omega^{-1/2} \lambda_x$  strongly as  $\kappa$  tends to 0. ■*

We substitute  $\{\lambda_x^\kappa\}$  for  $\{\lambda_x\}$  in our Hamiltonians and add a superscript  $\kappa$  to them. Clearly the cutoff full Hamiltonian converges no-cutoff one in the strong resolvent sense.

We introduce an operator  $S^\kappa := \sum_{x \in \Lambda} n_x \otimes \phi(i\omega^{-1} \lambda_x^\kappa)$ . Nelson's analytic vector theorem proves  $S^\kappa$  is self-adjoint. Hence we define a unitary transformation  $V^\kappa$  as follows [4]:

$$V^\kappa := e^{i\alpha S^\kappa}. \quad (2.17)$$

**Proposition. 2.3** [4] *The following expression holds:*

$$V^\kappa (1 \otimes H_b) (V^\kappa)^{-1} = 1 \otimes H_b + \alpha H_I + \alpha^2 R^\kappa \otimes 1, \quad (2.18)$$

where

$$R^\kappa := \frac{1}{2} \sum_{x, y \in \Lambda} \langle \omega^{-1/2} \lambda_x^\kappa, \omega^{-1/2} \lambda_y^\kappa \rangle n_x n_y. \quad (2.19)$$

(Proof) Note that  $\mathcal{D}_H$  is a total set of analytic vectors for  $S^\kappa$ , and is a core for  $1 \otimes H_b$  and  $S^\kappa$ . Let  $\Psi \in \mathcal{D}_H$  and  $\delta_{S^\kappa}(A) := [S^\kappa, A]$  ( $\delta_{S^\kappa}$  is a derivation). Since

$$V^\kappa (1 \otimes H_b) (V^\kappa)^{-1} \Psi = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \delta_{S^\kappa}^n (1 \otimes H_b) \Psi, \quad (2.20)$$

we will compute  $\delta_{S^\kappa}^n (1 \otimes H_b) \Psi$ . Remark that

$$[\phi(i\omega^{-1} \lambda_x^\kappa), H_b] \Psi = -i\phi(\lambda_x^\kappa) \Psi, \quad (2.21)$$

$$[\phi(i\omega^{-1} \lambda_x^\kappa), \phi(\lambda_y^\kappa)] \Psi = i\text{Im} \langle i\omega^{-1} \lambda_x^\kappa, \lambda_y^\kappa \rangle \Psi = -i \langle \omega^{-1/2} \lambda_x^\kappa, \omega^{-1/2} \lambda_y^\kappa \rangle \Psi. \quad (2.22)$$

From this it follows that

$$\delta_{S^\kappa}^0 (1 \otimes H_b) \Psi = 1 \otimes H_b \Psi, \quad (2.23)$$

$$\delta_{S^\kappa}^1 (1 \otimes H_b) \Psi = \sum_{x \in \Lambda} n_x \otimes [\phi(i\omega^{-1} \lambda_x^\kappa), H_b] \Psi = -i H_I \Psi, \quad (2.24)$$

$$\delta_{S^\kappa}^2 (1 \otimes H_b) \Psi = -2 R^\kappa \otimes 1 \Psi, \quad (2.25)$$

$$\delta_{S^\kappa}^n (1 \otimes H_b) \Psi = 0 \quad (n \geq 3). \quad (2.26)$$

Hence we obtain

$$V^\kappa (1 \otimes H_b) (V^\kappa)^{-1} \Psi = (1 \otimes H_b + \alpha H_I + \alpha^2 R^\kappa \otimes 1) \Psi. \quad (2.27)$$

As  $\Psi$  is an element of a core, the above equality holds in the operator sense. ■

Here we set an important

**Assumption. 2.3** For any  $x, y \in \Lambda$  and any  $\kappa > 0$ , put  $\langle \omega^{-1/2} \lambda_x^\kappa, \omega^{-1/2} \lambda_y^\kappa \rangle = \delta_{x,y} (b^\kappa)^2$ . ■

**Remark. 2.4** This assumption is not compatible with summability conditions in Remark 2.2 if the set  $\Lambda$  is infinite. ■

In the following we assume the above assumption. Defining

$$\hat{H}^\kappa := (V^\kappa)^{-1} (H_e \otimes 1) V^\kappa - (\alpha b^\kappa)^2 R^\kappa \otimes 1 + 1 \otimes H_b, \quad (2.28)$$

we have

$$H^\kappa = V^\kappa \hat{H}^\kappa (V^\kappa)^{-1}, \quad (2.29)$$

and hence  $H^\kappa$  is unitarily equivalent to  $\hat{H}^\kappa$ . Moreover it holds that

$$V^\kappa = e^{i\alpha S^\kappa} = \prod_{x \in \Lambda} e^{i\alpha n_x} \otimes e^{i\alpha \phi(i\omega^{-1} \lambda_x^\kappa)} = e^{i\alpha N_e} \otimes e^{i\alpha \sum_{x \in \Lambda} \phi(i\omega^{-1} \lambda_x^\kappa)}, \quad (2.30)$$

where  $N_e$  is the number operator for electrons. Since the Hubbard Hamiltonian  $H_e$  and  $N_e$  commute, we obtain

$$\hat{H}^\kappa = \hat{H}_e^\kappa \otimes 1 + 1 \otimes H_b, \quad (2.31)$$

$$\hat{H}_e^\kappa := d\Gamma_e(T) + \left( U - (\alpha b^\kappa)^2 \right) \sum_{x \in \Lambda} n_{x,+} n_{x,-} - \frac{(\alpha b^\kappa)^2}{2} N_e \quad (2.32)$$

$$= d\Gamma_e(\hat{T}) + \left( U - (\alpha b^\kappa)^2 \right) \sum_{x \in \Lambda} n_{x,+} n_{x,-}, \quad \hat{T} := T - \frac{(\alpha b^\kappa)^2}{2} 1. \quad (2.33)$$

Note that now we prove  $(\alpha b^\kappa)^2/2$  behaves like the chemical potential.

### 3 The property of a transformed Hamiltonian and a construction of a model under the infrared regular condition

We redefine  $\hat{T}$  as  $T$  and hence we can redefine  $\hat{H}_e^\kappa$  to the Hubbard type:

$$\hat{H}_e^\kappa = d\Gamma_e(T) + \left( U - (\alpha b^\kappa)^2 \right) \sum_{x \in \Lambda} n_{x,+} n_{x,-}. \quad (3.1)$$

From here we fix a number of electrons  $N_e$  (we consider  $N_e$  a number not an operator). Denote  $\mathcal{H}_e$  as an  $N_e$  particle subspace of  $\mathcal{F}_e$ . In the following our full Hilbert space is  $\mathcal{H}_e \otimes \mathcal{F}_b$ .

Since  $\hat{H}^\kappa$  has no interaction between electrons and phonons, and since the behavior of  $1 \otimes H_b$  is well-known, all we have to do is the analysis of (ground states of)  $\hat{H}_e^\kappa$ . It is notable for  $\hat{H}_e^\kappa$  that interaction between electrons becomes  $(U - (\alpha b^\kappa)^2)$ : it is attractive for sufficiently large  $\alpha$ .

Now the existence of ground states is trivial for  $\hat{H}^\kappa$ . Thus in the following it suffice to argue with the uniqueness problem. We show the sign of  $(U - (\alpha b^\kappa)^2)$  changes the uniqueness property. We must set assumptions and define various physical objects. Let us start with the case  $U - (\alpha b^\kappa)^2 < 0$ .

**Assumption. 3.1** The hopping matrix  $T = (t_{x,y})$  is real symmetric. ■

We have a *bond* between sites  $x$  and  $y$  if  $t_{x,y}$  does not vanish, and  $\Lambda$  is said to be *connected* if there is a connected path of bonds between every pair of sites.

**Assumption. 3.2**  $\Lambda$  is connected. ■

We introduce the total spin operators  $\hat{S}_{\text{tot}} = (\hat{S}_{\text{tot}}^{(1)}, \hat{S}_{\text{tot}}^{(2)}, \hat{S}_{\text{tot}}^{(3)})$  of the Hubbard system by

$$\hat{S}_{\text{tot}}^{(i)} := \frac{1}{2} \sum_{x,y \in \Lambda, s,t=\pm} c_{x,s}^* \sigma_{s,t}^{(i)} c_{y,t}, \quad i = 1, 2, 3, \quad (3.2)$$

where  $\sigma^{(i)}$  are the Pauli matrices. We denote by  $S_{\text{tot}}(S_{\text{tot}} + 1)$  the eigenvalue of  $(\hat{S}_{\text{tot}})^2 := \sum_{i=1}^3 (\hat{S}_{\text{tot}}^{(i)})^2$ . We call a quantity  $S_{\text{tot}} \geq 0$  the total spin of the state. Let

$$S_{\text{max}} = \begin{cases} \frac{1}{2} N_e, & N_e \leq |\Lambda|; \\ \frac{1}{2} (2|\Lambda| - N_e), & N_e > |\Lambda|, \end{cases} \quad (3.3)$$

where  $N_e$  is a number of electrons (natural number). This is the maximum value of  $S_{\text{tot}}$ .

**Theorem. 3.1** [20] Assume  $U - (\alpha b^\kappa)^2 \leq 0$  and that  $N_e$  is even. Then

1. among the ground states of  $\hat{H}_e^\kappa$  there is one with total spin  $S_{\text{tot}} = 0$ ;
2. if  $U - (\alpha b^\kappa)^2 < 0$ , the ground state is unique and hence has  $S_{\text{tot}} = 0$ . ■

From this theorem we can construct models whose ground state is unique and spin-singlet if  $U - (\alpha b^\kappa)^2 < 0$ . It is remarkable to allow various choices for  $T$  and  $\Lambda$ . We shall see further assumptions break the uniqueness of the ground states for the case  $U - (\alpha b^\kappa)^2 > 0$ . Before stating the theorem we need a

**Definition. 3.2** A Hubbard model exhibits ferromagnetism if all ground states have a total spin  $S_{\text{tot}} = S_{\text{max}}$ . ■

**Theorem. 3.3** [28](Theorem 5.1) We associate with each site  $x$  a constant  $t_x > 0$ , and the hopping matrix  $T = (t_{x,y})$  with long range hopping amplitudes by

$$t_{x,y} = t_0 t_x t_y \quad (x, y \in \Lambda) \quad (3.4)$$

where  $t_0 > 0$  is a constant. If the electron number is  $N_e = |\Lambda| - 1$  and  $U - (\alpha b^\kappa)^2 > 0$ , then the model exhibits ferromagnetism and the ground states are non-degenerate apart from the trivial  $(2S_{\text{max}} + 1)$ -fold degeneracy. ■

**Remark. 3.4** The electrons whose hopping matrix is  $t_{x,y} = t_0 t_x t_y$  can hop from any site in the lattice to any other site, and there is only one hole. These assumptions are unphysical. See [28] for more physically acceptable ferromagnetic Hubbard models. ■

Since the conditions of two theorems are compatible with assumptions and since the electron number becomes even by taking suitable  $\Lambda$ , we can construct examples which satisfy the condition of the above two theorems simultaneously.

## 4 Removal of infrared cutoff

In this section we remove infrared cutoff. This section is essentially the copy of the argument in the van Hove model [2]. However, for the reader's sake, we outline it.

By the same argument as in Proposition 2.3, the following expression holds.

**Theorem. 4.1** Under the assumptions in section 2 and 3 we obtain

$$V^\kappa (1 \otimes N_b) (V^\kappa)^{-1} = 1 \otimes N_b + \alpha \sum_{x \in \Lambda} n_x \otimes \phi(\omega^{-1} \lambda_x^\kappa) + \alpha^2 \sum_{x \in \Lambda} n_x^2 \otimes 1 \|\omega^{-1} \lambda_x^\kappa\|^2, \quad (4.1)$$

where  $N_b$  is the number operator for bosons. ■

In the previous section we show that  $\hat{H}^\kappa$  and  $H^\kappa$  have ground states of the form

$$\hat{\Psi}_g^\kappa := \Psi_{e,g}^\kappa \otimes \Omega_b, \quad (4.2)$$

$$\Psi_g^\kappa := V^\kappa \hat{\Psi}_g^\kappa, \quad (4.3)$$

where  $\Psi_{e,g}^\kappa$  is an arbitrary normalized ground state of the Hubbard model  $H_e^\kappa$  and  $\Omega_b$  is the bosonic Fock vacuum. Hence we have the following estimate,

$$\langle \Psi_g^\kappa, 1 \otimes N_b \Psi_g^\kappa \rangle = \alpha^2 \sum_{x \in \Lambda_e} \|\omega^{-1} \lambda_x^\kappa\|^2, \quad (4.4)$$

where  $\Lambda_e$  is a set of sites on which electrons exist in a state vector. The RHS diverges as  $\kappa$  tends to 0 if  $\omega$  is massless due to the Assumption 2.1.

**Remark. 4.2** Consider the case  $|\Lambda| = \infty$  and  $N_e = \infty$  formally. Then we face infrared divergence when the sequence  $\{\|\omega^{-1} \lambda_x^\kappa\|\}$  is not summable even if phonons are optical (massive). Furthermore this summability breaks down the translation invariance of the Hubbard model  $H_e^\kappa$  and  $\hat{H}_e^\kappa$ .

We would like to construct a suitable representation theory. Before that we investigate interesting properties of the Fock representation under the infrared singular condition.

Denote

$$\tilde{H}^\kappa := H^\kappa - E_0(H^\kappa), \quad (4.5)$$

$$a_{\omega,\kappa}(f) := V^\kappa (1 \otimes a(f)) (V^\kappa)^{-1} = 1 \otimes a(f) + \frac{\alpha}{\sqrt{2}} \sum_{x \in \Lambda} \langle \omega^{-1/2} f, \omega^{-1/2} \lambda_x^\kappa \rangle n_x \otimes 1, \quad f \in \text{dom } \omega^{-1/2}, \quad (4.6)$$

where  $a(f)$  is an annihilator of bosons.

**Theorem. 4.3** For any  $t \in \mathbb{R}$ ,  $f \in \text{dom } \omega^{-1/2}$ ,  $\Psi \in \text{dom } 1 \otimes H_b^{1/2}$ , we have equalities

$$e^{it\tilde{H}^\kappa} a_{\omega,\kappa}(f) e^{-it\tilde{H}^\kappa} \Psi = a_{\omega,\kappa}(e^{it\omega} f) \Psi, \quad (4.7)$$

$$e^{it\tilde{H}^\kappa} a_{\omega,\kappa}^*(f) e^{-it\tilde{H}^\kappa} \Psi = a_{\omega,\kappa}^*(e^{it\omega} f) \Psi, \quad (4.8)$$

$$a_{\omega,\kappa}(f) \Psi_g^\kappa = 0, \quad \forall \kappa > 0. \quad (4.9)$$

**Remark. 4.4** The operator  $a_{\omega,\kappa}(f)$  annihilates the ground states  $\Psi_g^\kappa$  of  $H^\kappa$ .

(Proof of Theorem) Let  $\Psi, \Phi \in \text{dom } H^\kappa$ ,  $f \in \text{dom } \omega^{-1/2} \cap \text{dom } \omega$ , and

$$v(t) := \langle \Phi, e^{it\tilde{H}^\kappa} a_{\omega,\kappa}(e^{-it\omega} f) e^{-it\tilde{H}^\kappa} \Psi \rangle. \quad (4.10)$$

Using the relation

$$\left[ 1 \otimes a(f), \tilde{H}^\kappa \right] = 1 \otimes a(\omega f) + \frac{\alpha}{\sqrt{2}} \sum_{x \in \Lambda} \langle f, \lambda_x^\kappa \rangle n_x \otimes 1 = a_{\omega,\kappa}(\omega f), \quad (4.11)$$

we obtain

$$\frac{dv(t)}{dt} = 0 \implies e^{it\tilde{H}^\kappa} a_{\omega,\kappa}(e^{-it\omega} f) e^{-it\tilde{H}^\kappa} = a_{\omega,\kappa}(f) \Psi. \quad (4.12)$$

Setting  $f$  to  $e^{it\omega} f$ , we have the result under the above condition. For full equality, we use limiting argument. Third equality follows from definitions. ■

**Lemma. 4.5** Suppose  $\omega$  is absolutely continuous. If  $f \in \text{dom } \omega^{-1/2}$ ,  $\Psi \in \text{dom } 1 \otimes H_b^{1/2}$  then

$$\lim_{t \rightarrow \pm\infty} 1 \otimes a(e^{it\omega} f) \Psi = 0. \quad (4.13)$$

(Proof) Set

$$\Psi = 1 \otimes a(f_1)^* \cdots 1 \otimes a(f_n)^* \Psi_e \otimes \Omega_b, \quad \Psi_e \in \mathcal{H}_e. \quad (4.14)$$

From this equality we have

$$1 \otimes a(e^{it\omega} f) \Psi = \sum_{j=1}^n \langle e^{it\omega} f, f_j \rangle 1 \otimes a(f_1)^* \cdots 1 \otimes \widehat{a(f_j)^*} \cdots 1 \otimes a(f_n)^* \Psi_e \otimes \Omega_b, \quad (4.15)$$

where  $1 \otimes \widehat{a(f)^*}$  means it is removed. Since  $\omega$  is absolutely continuous the above expression tends to 0 as  $t \rightarrow \pm\infty$ .

Note that  $\mathcal{H}_e \hat{\otimes} \mathcal{F}_{b,\text{fin}}(\text{dom } \omega)$  is a core of  $\text{dom } 1 \otimes H_b^{1/2}$ . It follows that, for any  $\Psi \in \text{dom } 1 \otimes H_b^{1/2}$ , there exist vectors  $\Psi_n \in \mathcal{H}_e \hat{\otimes} \mathcal{F}_{b,\text{fin}}(\text{dom } \omega)$  such that

$$\Psi_n \rightarrow \Psi, \quad 1 \otimes H_b^{1/2} \Psi_n \rightarrow 1 \otimes H_b^{1/2} \Psi. \quad (4.16)$$

Hence we obtain

$$\|1 \otimes a(e^{it\omega} f) \Psi\| \leq \|\omega^{-1/2} f\| \|1 \otimes H_b^{1/2} (\Psi - \Psi_n)\| + \|1 \otimes a(e^{it\omega} f) \Psi_n\| \quad (4.17)$$

$$\implies \limsup_{t \rightarrow \pm\infty} \|1 \otimes a(e^{it\omega} f) \Psi\| \leq \|\omega^{-1/2} f\| \|1 \otimes H_b^{1/2} (\Psi - \Psi_n)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \blacksquare \quad (4.18)$$

**Theorem. 4.6** If  $\omega$  is absolutely continuous  $H$  has no point spectrum, in particular no ground states.

(Proof) Assume  $E \in \sigma_p(H)$ ,  $\Psi_E = (\Psi_E^{(n)})_{n \geq 0} \neq 0$  is an eigenvector for  $E$  and  $f \in \text{dom } \omega^{-1/2}$ . By Theorem 4.3

$$e^{it(H-E)} a_{\omega,\kappa}(f) \Psi_E = a_{\omega,\kappa}(e^{it\omega} f) \Psi_E. \quad (4.19)$$

Since due to Lemma 4.5 we have

$$\|a_{\omega,\kappa}(f) \Psi_E\| = \|a_{\omega,\kappa}(e^{it\omega} f) \Psi_E\| \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty, \quad (4.20)$$

it follows that

$$1 \otimes a(f)\Psi_E = -\frac{\alpha}{\sqrt{2}} \sum_{x \in \Lambda} \langle \omega^{-1/2} f, \omega^{-1/2} \lambda_x \rangle n_x \otimes 1\Psi_E. \quad (4.21)$$

Since  $\Psi_E \neq 0$  there exists an  $n \in \mathbb{N}$  such that  $c_n := \|\Psi_E^{(n)}\|^2 \neq 0$ . Thus we obtain

$$\langle \Psi_E^{(n)}, 1 \otimes a(f)\Psi_E \rangle = -\frac{\alpha c_n}{\sqrt{2}} \sum_{x \in \Lambda_e} \langle \omega^{-1/2} f, \omega^{-1/2} \lambda_x \rangle. \quad (4.22)$$

Putting  $F(f) := \langle 1 \otimes a(f)\Psi_E, \Psi_E^{(n)} \rangle$ ,  $f \in \text{dom } \omega^{-1/2}$ , we have

$$|F(f)| \leq \|1 \otimes a(f)^* \Psi_E^{(n)}\| \|\Psi_E\| \leq \|f\| \|(1 \otimes N_b + 1)^{1/2} \Psi_E^{(n)}\| \|\Psi_E\| \leq \sqrt{n+1} \|f\| \|\Psi_E\|^2. \quad (4.23)$$

Riesz's representation theorem asserts that there uniquely exists a vector  $u \in \mathcal{K}$  such that

$$F(f) = \langle u, f \rangle = -\frac{\alpha c_n}{\sqrt{2}} \langle \omega^{-1/2} \sum_{x \in \Lambda_e} \lambda_x, \omega^{-1/2} f \rangle. \quad (4.24)$$

This leads  $(\sum_{x \in \Lambda_e} \lambda_x) \in \text{dom } \omega^{-1}$  and hence contradicts the Assumption 2.3 ■

**Theorem. 4.7**  $\Psi_g^\kappa$  converges to 0 weakly as  $\kappa$  tends to 0.

(Proof) Suppose  $\Psi_e$  and  $f_i$  are arbitrary vectors in  $\mathcal{H}_e$  and  $\mathcal{K}$ , and  $\Psi_g^\kappa$  is a ground state of  $H^\kappa$ . Since  $\Psi_g^\kappa = V^\kappa \Psi_{e,g}^\kappa \otimes \Omega_b$  we prove an equality

$$\langle 1 \otimes a(f_1)^* \cdots 1 \otimes a(f_n)^* \Psi_e \otimes \Omega_b, \Psi_g^\kappa \rangle = e^{i\alpha N_e} \left( \frac{-1}{\sqrt{2}} \right)^n e^{-\frac{\alpha^2}{4} \sum_{x \in \Lambda_e} \|\omega^{-1} \lambda_x^\kappa\|^2} \prod_{i=1}^n \left( \sum_{x \in \Lambda_e} \langle f_i, \omega^{-1} \lambda_x^\kappa \rangle \right) \langle \Psi_e, \Psi_{e,g}^\kappa \rangle. \quad (4.25)$$

Then it follows that

$$|\langle 1 \otimes a(f_1)^* \cdots 1 \otimes a(f_n)^* \Psi_e \otimes \Omega_b, \Psi_g^\kappa \rangle| \quad (4.26)$$

$$\leq 2^{-n/2} e^{-\frac{\alpha^2}{4} \sum_{x \in \Lambda_e} \|\omega^{-1} \lambda_x^\kappa\|^2} \prod_{i=1}^n \left( \|f_i\| \sum_{x \in \Lambda_e} \|\omega^{-1} \lambda_x^\kappa\| \right) \|\Psi_e\| \rightarrow 0 \text{ as } \kappa \rightarrow 0. \quad (4.27)$$

From the fact  $\sup_{\kappa > 0} \|\Psi_g^\kappa\| = 1$  and the Assumption 2.1 the statement holds. ■

From this theorem we cannot use Hilbert space techniques for removal of infrared cutoff; otherwise we use the operator algebraic technique.

**Remark. 4.8** We can compute the Wightman functionals concretely. See [2].

Define the following  $C^*$ -algebras and dynamics:

$$\mathcal{A}^{\text{irr}} := \mathcal{A}_e \otimes \mathcal{A}_b^{\text{irr}}, \quad (4.28)$$

$$\mathcal{A}^{\text{irs}} := \mathcal{A}_e \otimes \mathcal{A}_b^{\text{irs}}, \quad (4.29)$$

$$\mathcal{A}_e := \mathbb{B} \left( \bigotimes_{\text{as}}^{N_e} \ell^2(\Lambda; \mathbb{C}^2) \right), \quad (4.30)$$

$$\mathcal{A}_b^{\text{irr}} := C^* \left\{ W^{\text{irr}}(f) : f \in \text{dom } \omega^{-1/2} \right\}, \quad W^{\text{irr}}(f) := e^{i\phi(f)}, \quad (4.31)$$

$$\mathcal{A}_b^{\text{irs}} := C^* \left\{ W^{\text{irs}}(f) : f \in \text{dom } \omega^{-1/2} \right\}, \quad W^{\text{irs}}(f) := e^{i\phi^{\text{irs}}(f)}, \quad \phi^{\text{irs}}(f) := \phi(f) - \alpha \sum_{x \in \Lambda_e} \langle \omega^{-1/2} \lambda_x, \omega^{-1/2} f \rangle, \quad (4.32)$$

$$\alpha_t^\kappa(A) := e^{itH^\kappa} A e^{-itH^\kappa}, \quad \kappa > 0, \quad (4.33)$$

$$\alpha_t^0(A) := e^{it\hat{H}^0} A e^{-it\hat{H}^0}, \quad (4.34)$$

where  $\mathbb{B}(\mathcal{H})$  is a  $C^*$ -algebra of all bounded operators on a Hilbert space  $\mathcal{H}$  and  $C^*\{\cdot\}$  means a  $C^*$ -closure of a set  $\{\cdot\}$ . Here we abuse notation for superscripts  $\kappa \geq 0$  but we can easily justify the expressions for  $\kappa = 0$ . Then the state  $\psi_g^\kappa(A) = \langle \Psi_g^\kappa, A \Psi_g^\kappa \rangle$ ,  $A \in \mathcal{A}^{\text{irr}}$  is a ground state for the dynamics  $\alpha_t^\kappa$  (for  $\kappa > 0$ ).



Set  $A^{\text{irr}} = A_e \otimes W^{\text{irr}}(f)$  and  $A^{\text{irs}} = A_e \otimes W^{\text{irs}}(f)$ . It follows that

$$\psi_g^\kappa(A^{\text{irr}}) = e^{-i\alpha \text{Re}\langle \omega^{-1/2} \sum_{x \in \Lambda_e} \lambda_x^\kappa, \omega^{-1/2} f \rangle} \langle \Psi_{e,g}^\kappa \otimes \Omega_b, A^{\text{irr}} \Psi_{e,g}^\kappa \otimes \Omega_b \rangle \quad (4.35)$$

$$\rightarrow e^{-i\alpha \text{Re}\langle \omega^{-1/2} \sum_{x \in \Lambda_e} \lambda_x, \omega^{-1/2} f \rangle} \langle \Psi_{e,g}^0 \otimes \Omega_b, A^{\text{irr}} \Psi_{e,g}^0 \otimes \Omega_b \rangle \quad (4.36)$$

$$= \langle \Psi_{e,g}^0 \otimes \Omega_b, A^{\text{irs}} \Psi_{e,g}^0 \otimes \Omega_b \rangle =: \psi_g^0(A^{\text{irs}}) \quad \kappa \rightarrow 0. \quad (4.37)$$

Since the above argument leads weak\* convergence of a state and since the dynamics  $\alpha_t^\kappa$  also converges to  $\alpha_t^0$ , we obtain the following

**Theorem. 4.9**  $(\mathcal{A}^{\text{irs}}, \psi_g^0, \alpha_t^0)$  is a limit representation of  $(\mathcal{A}^{\text{irr}}, \psi_g^\kappa, \alpha_t^\kappa)$  on  $\mathcal{F}$  and defines a theory under infrared singular condition.

In the representation  $(\mathcal{A}^{\text{irs}}, \psi_g^0, \alpha_t^0)$  the total Hamiltonian is  $\hat{H}^0 - E_0(\hat{H}^0)$ . The representation of CAR is the Fock representation. Hence we can apply the analysis in section 3: we can construct ferromagnetic or unique spin-singlet ground states under the *infrared singular condition*.

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